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# Observable Correlations in Two-Qubit States

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**Abstract** The total correlations in a bipartite quantum state are well quantified by the quantum mutual information, the amount of which is not necessarily fully extractable by local measurements. The observable correlations are the maximal correlations that can be extracted via local measurements, and have an intuitive interpretation as a measure of classical correlations. We evaluate the observable correlations for generic two-qubit states and obtain analytical expressions in some particular cases. The intricate and subtle relationships among the total, quantum and classical correlations are illustrated in terms of observable correlations. In the course, we also disprove an intuitive conjecture of Lindblad which states that the classical correlations account for at least half of the total correlations, or equivalently, correlations are more classical than quantum.

**Keywords** Quantum mutual information · Von Neumann measurement · Observable correlations · Classical correlations · Lindblad conjecture

## 1 Introduction

Correlations are a recurring theme in science, in particular, in statistics and quantum physics. In the classical scenario, correlations are usually characterized by covariance matrices, various correlation coefficients [7], and entropic quantities such as the Shannon mutual information [6, 23].

In the quantum scenario, correlations gain a new dimension due to entanglement and non-commutativity [8, 18, 21]. The so-called quantum correlations serve as a valuable resource for various computation and communications tasks [18], and one is naturally led to

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inquire the relationships between classical and quantum correlations. Quantum correlations are usually studied in the entanglement/separability paradigm first formalized by Werner [26]. This framework has been extensively developed in the last decade in connection with quantum information and quantum computation [13, 27].

In this article, pursuing the original ideas of Everett and Lindblad [9, 15, 17], and putting them into the context of modern quantum information theory, we will study correlations (both classical and quantum) from a measurement perspective. Our main results are the analytic formulas for the observable correlations of some two-qubit states. As an application, we disprove an intuitive conjecture of Lindblad which states roughly that correlations are more classical than quantum [17].

The central characters here will be quantum states, which are mathematically represented by non-negative operators with unity trace (i.e., density operators), and von Neumann measurements, which are represented by complete sets of orthogonal one-dimensional projections (resolution of the identity) [20]. Consider a bipartite quantum state $\rho$ shared by parties $a$ and $b$ with marginal states $\rho^a = \mathrm{tr}_b\rho$, $\rho^b = \mathrm{tr}_a\rho$ (partial trace), one is interested in how many correlations it can encode. The correlations are often classified into three categories: total, quantum and classical, and a basic issue in quantum information theory is to quantify them.

For the total correlations, it is well quantified by the quantum mutual information [1–3, 10, 12, 14, 15, 19, 22, 24]

$$\mathcal{I}(\rho) = S(\rho^a) + S(\rho^b) - S(\rho)$$

where $S(\rho) = -\mathrm{tr}\rho\log\rho$ is the von Neumann entropy (the logarithm is understood in base 2 in this article) [24, 25]. Recently, Groisman et al. [10] and Schumacher and Westmoreland [22] have presented very appealing and significant arguments for regarding the quantum mutual information as a measure of total correlations.

One naturally wants to inquire, among the total correlations $\mathcal{I}(\rho)$, how many can be regarded as classical, and how many can be regarded as quantum. Of course, one cannot expect a unique solution for such a problem, and there are several approaches to this issue, such as that based on entanglement [4, 13, 27], on quantum communications [12], and on quantum discord [19].

Here we will pursue the approach based on local quantum measurements, which originated from the earlier idea and work of Everett and Lindlad [9, 15], and is further investigated by Hall et al. [11]. It is truly remarkable that, well before the emergence of quantum information theory, Everett initiated an informational approach to quantum mechanics as early as 1957, in which the Shannon information theory and the Schmidt representation play a basic role, and Lindblad studied extensively the relationships between quantum measurement and information, in particular the relationships between the quantum mutual information and its local measurement realization [15–17].

Note that the quantum mutual information $\mathcal{I}(\rho)$ is a theoretical quantity independent of measurements, and a natural question arises: How many correlations can be extracted from $\rho$ if both parties $a$ and $b$ perform only local measurements? By a measurement here we mean the von Neumann type (complete measurement consisting of orthogonal one-dimensional projections).

Let $\{P_j^a\}$ and $\{P_k^b\}$ be local measurements pertinent to parties $a$ and $b$, respectively, then after the measurement, the state $\rho$ changes to

$$P(\rho) = \sum_{jk} P_j^a \otimes P_k^b \rho P_j^a \otimes P_k^b.$$

This final state is essentially a quantum formalism of the classical bivariate probability distribution $p=\{p_{jk}\}$ with

$$p_{jk}=\mathrm{tr}P_j^a\otimes P_k^b\rho P_j^a\otimes P_k^b.$$

Indeed (noting that $P_j^a$ and $P_k^b$ are one-dimensional projections), we have

$$P(\rho)=\sum_{jk}p_{jk}P_j^a\otimes P_k^b,$$

and the total correlations in $P(\rho)$ is

$$\mathcal{I}(P(\rho))=I(p)=H(p^a)+H(p^b)-H(p)$$

which is actually the amount of correlations obtained via the measurement $P=\{P_j^a\otimes P_k^b\}$. Here $p_j^a=\sum_k p_{jk},\ p_k^b=\sum_j p_{jk},\ H(\cdot)$ is the Shannon entropy functional, e.g., $H(p)=-\sum_{jk}p_{jk}\log p_{jk}$, and $I(p)$ is the classical mutual information [6, 23].

By optimizing over local measurements $P$, one is led to the observable correlations

$$\mathcal{C}(\rho)=\sup_P\mathcal{I}(P(\rho)).$$

Since this quantity is the maximally extractable correlations via *local* measurements, it can be interpreted as a measure of classical correlations. This measure of correlations has been extensively studied by Lindblad [15, 17] who, based on several intuitive observations, further proposed the following conjecture [17]

$$\mathcal{C}(\rho)\ge\frac{1}{2}\mathcal{I}(\rho),$$

which remains open until now.

To gain some intuition about the above conjecture, let us consider some extreme cases.

First, let $\rho=|\Psi\rangle\langle\Psi|$ be a pure bipartite state, then it can always be written in the Schmidt form as [18]

$$|\Psi\rangle=\sum_j\sqrt{\lambda_j}|\psi_j^a\rangle\otimes|\psi_j^b\rangle.$$

Here $\{|\psi_j^a\rangle\}$ and $\{|\psi_j^b\rangle\}$ are orthonormal sets for parties $a$ and $b$, respectively. Direct calculations yield [11]

$$\mathcal{I}(\rho)=2S,\qquad\mathcal{C}(\rho)=S$$

where $S=-\sum_j|\lambda_j|\log|\lambda_j|$ is the reduced von Neumann entropy, which turns out also to be the conventional entanglement of the pure state $\rho$ [4]. Therefore, we see that for any pure state, the amount of its classical correlations is precisely half of the amount of total correlations, and the other half can be regarded as quantum correlations. An interesting and significant operational meaning of this equal distribution of correlations is proposed by Groisman et al. [10].

Second, consider a classical bipartite state $p=\{p_{jk}\}$, which is a classical bivariate probability distribution. We can write it in the quantum formalism as

$$\rho=\sum_{jk}p_{jk}P_j^a\otimes P_k^b$$

where $\{P_j^a\}$ and $\{P_k^b\}$ are sets of orthogonal projections for parties $a$ and $b$, respectively. Then direct evaluation yields

$$\mathcal{C}(\rho) = \mathcal{I}(\rho) = I(p).$$

Thus in this case, all correlations are classical, and the quantum correlations have to vanish, which is just what our intuition requires.

Consequently, we see that the Lindblad conjecture is true in the above two extreme cases. For the intermediate cases, a plausible argument supporting the Lindblad conjecture is that the intermediate states are classical mixtures of pure quantum states, and one might expect that quantum correlations decrease and classical correlations increase after this mixing process.

It is usually difficult and even intractable to evaluate the observable correlations due to the complicated optimization procedures involved. In this article, we derive the analytical expressions of the observable correlations for a class of two-qubit states in Sect. 2, and use them to illustrate various relationships among the total, quantum and classical correlations in Sect. 3. Incidentally, we disprove the Lindblad conjecture, and show that counterexamples abound. Finally, in Sect. 4, we make some further comments and discussions about the relationships among the observable correlations, the measure of classical correlations introduced by Henderson and Vedral [12], and the quantum discord introduced by Ollivier and Zurek [19].

## 2 Two-Qubit States

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli spin matrices acting on $C^2$. Because $\{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3\}$ constitutes an operator base for the space of all operators on $C^2$, any two-qubit state can be written as

$$w = \frac{1}{4}\left(\mathbf{1} + \vec{\alpha}\vec{\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{\beta}\vec{\sigma} + \sum_{j,k=1}^{3} \gamma_{jk}\sigma_j \otimes \sigma_k\right).$$

Here $\mathbf{1}$ is the identity operator on the composite system or on the component systems, depending on the context, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\vec{\beta} = (\beta_1, \beta_2, \beta_3) \in R^3$; $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^{\mathrm{T}}$ (here T denotes transposition) is written as a column vector with each component being the corresponding Pauli operator, $\vec{\alpha}\vec{\sigma} = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$, etc., and $\gamma_{jk}$ are real numbers.

Our aim is to evaluate the observable correlations of the above state. But first, without loss of generality, we will make some reduction by means of the local unitary invariance of the observable correlations.

Any unitary matrix $U \in$SU(2) can be represented as

$$U = s\mathbf{1} + i\vec{u}\vec{\sigma}$$

with $s \in R, \vec{u} = (u_1, u_2, u_2) \in R^3$ and $s^2 + u_1^2 + u_2^2 + u_3^2 = 1$. The following results can be verified by direct calculations.

**Lemma 1** *Let $U = s\mathbf{1} + i\vec{u}\vec{\sigma} \in SU(2)$ be a unitary matrix, and $O$ be the matrix defined by*

$$O = \begin{pmatrix} s^2+u_1^2-u_2^2-u_3^2 & 2(su_3+u_1u_2) & 2(-su_2+u_1u_3) \\ 2(-su_3+u_1u_2) & s^2+u_2^2-u_1^2-u_3^2 & 2(su_1+u_2u_3) \\ 2(su_2+u_1u_3) & 2(-su_1+u_2u_3) & s^2+u_3^2-u_1^2-u_2^2 \end{pmatrix},$$

*then $O$ is an orthogonal matrix, and moreover*

$$U^\dagger(\vec{\alpha}\vec{\sigma})U = \vec{\alpha}O\vec{\sigma}, \quad \forall \vec{\alpha} \in R^3.$$

*Conversely, for any orthogonal matrix $O \in SO(3)$, there exists a unitary matrix $U \in SU(2)$ such that the above equation holds for any vector $\vec{\alpha}$.*

According to the singular value decomposition theorem [5], the matrix $\Gamma = \{\gamma_{jk}\}$ can always be written as

$$\Gamma = O^a \mathrm{diag}\{c_1, c_2, c_3\} O^b$$

with $O^a = \{O^a_{jk}\}$ and $O^b = \{O^b_{jk}\}$ being orthogonal matrices in SO(3). Consequently, $(O^a)^{\mathrm{T}}\Gamma(O^b)^{\mathrm{T}} = \mathrm{diag}\{c_1, c_2, c_3\}$, or more explicitly,

$$\sum_{j,k=1}^{3} \gamma_{jk} O^a_{jm} O^b_{nk} = c_m \delta_{m,n}.$$

Now by Lemma 1, there exist unitary matrices $U_a$ and $U_b$ such that

$$U_a^\dagger(\vec{\alpha}\vec{\sigma})U_a = \vec{\alpha}O_a\vec{\sigma}, \qquad U_b^\dagger(\vec{\beta}\vec{\sigma})U_b = \vec{\beta}O_b\vec{\sigma}.$$

Consequently,

$$U_a^\dagger \otimes U_b^\dagger w U_a \otimes U_b = \frac{1}{4}\left(\mathbf{1} + \vec{a}\vec{\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{b}\vec{\sigma} + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j\right)$$

where $\vec{a} = \vec{\alpha}O_a$, $\vec{b} = \vec{\beta}O_b$. Therefore we have the following result.

**Lemma 2** *Any two-qubit state, up to local unitary equivalence, can be represented as*

$$\rho = \frac{1}{4}\left(\mathbf{1} + \vec{a}\vec{\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{b}\vec{\sigma} + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j\right). \tag{1}$$

From the definition, the observable correlations is locally unitary invariant in the sense that

$$\mathcal{C}(U \otimes V w U^\dagger \otimes V^\dagger) = \mathcal{C}(w)$$

for any $U, V \in$SU(2). Therefore, without loss of any generality, we only need to evaluate $\mathcal{C}(\rho)$ with $\rho$ defined by (1).

**Theorem 1** *Let $\rho$ be defined by (1), then its observable correlations are given by*

$$\mathcal{C}(\rho) = \sup_{\vec{x},\vec{y}} I(p).$$

*Here the sup is over all vectors* $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3) \in R^3$ *satisfying*

$$x_1^2 + x_2^2 + x_3^2 = 1, \qquad y_1^2 + y_2^2 + y_3^2 = 1,$$

*and* $p = \{p_{jk}\}$ *is given by*

$$p_{00} = \frac{1}{4}(1 + \vec{a}\vec{x} + \vec{b}\vec{y} + \vec{c}\vec{x}\vec{y}),$$

$$p_{01} = \frac{1}{4}(1 + \vec{a}\vec{x} - \vec{b}\vec{y} - \vec{c}\vec{x}\vec{y}),$$

$$p_{10} = \frac{1}{4}(1 - \vec{a}\vec{x} + \vec{b}\vec{y} - \vec{c}\vec{x}\vec{y}),$$

$$p_{11} = \frac{1}{4}(1 - \vec{a}\vec{x} - \vec{b}\vec{y} + \vec{c}\vec{x}\vec{y})$$

*where* $\vec{c}\vec{x}\vec{y} = \sum_{j=1}^{3} c_j x_j y_j$.

Since further expression of $\mathcal{C}(\rho)$ will involve complicated implicit transcendental equations, we will not pursue them here, and will rather consider some particular cases with analytical solutions in order to illustrate some intrigue relations between the classical and quantum correlations.

We first establish Theorem 1. let $\{\Pi_j^a = |j\rangle\langle j| : j = 0, 1\}$ and $\{\Pi_k^b = |k\rangle\langle k| : k = 0, 1\}$ be the local measurements for parties $a$ and $b$ along their respective computational bases, then any general local von Neumann measurements can be written as

$$P_j^a = U\Pi_j^a U^\dagger, \qquad P_k^b = V\Pi_k^b V^\dagger$$

for some unitary matrices $U, V \in \mathrm{SU}(2)$. Note that any general unitary matrices $U$ and $V$ can be written, up to constant phases, as

$$U = s\mathbf{1} + i\vec{u}\vec{\sigma}, \qquad V = t\mathbf{1} + i\vec{v}\vec{\sigma}$$

with $s, t \in R$, $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in R^3$, and

$$s^2 + u_1^2 + u_2^2 + u_3^2 = 1, \qquad t^2 + v_1^2 + v_2^2 + v_3^2 = 1.$$

After the measurement $P = P^a \otimes P^b$, the state $\rho$ changes to

$$P(\rho) = \sum_{jk} P_j^a \otimes P_k^b \rho P_j^a \otimes P_k^b.$$

Now

$$\begin{aligned}
&(U^\dagger \otimes V^\dagger)P(\rho)(U \otimes V) \\
&\quad = (U^\dagger \otimes V^\dagger)\sum_{jk}(P_j^a \otimes P_k^b)\rho(P_j^a \otimes P_k^b)(U \otimes V) \\
&\quad = (U^\dagger \otimes V^\dagger)\sum_{jk}(U\Pi_j^a U^\dagger \otimes V\Pi_k^b V^\dagger)\rho(U\Pi_j^a U^\dagger \otimes V\Pi_k^b V^\dagger)(U \otimes V) \\
&\quad = \sum_{jk}(\Pi_j^a \otimes \Pi_k^b)(U^\dagger \otimes V^\dagger)\rho(U \otimes V)(\Pi_j^a \otimes \Pi_k^b).
\end{aligned}$$

Due to local unitary invariance, we have

$$\mathcal{I}(P(\rho)) = \mathcal{I}(\tau) = I(p)$$

where

$$\tau = \sum_{jk} (\Pi_j^a \otimes \Pi_k^b)(U^\dagger \otimes V^\dagger)\rho(U \otimes V)(\Pi_j^a \otimes \Pi_k^b)$$
$$= \sum_{jk} p_{jk} \Pi_j^a \otimes \Pi_k^b$$

with

$$p_{jk} = \mathrm{tr}(\Pi_j^a \otimes \Pi_k^b)(U^\dagger \otimes V^\dagger)\rho(U \otimes V)(\Pi_j^a \otimes \Pi_k^b).$$

Thus we only need to evaluate $I(p)$ for $p = \{p_{jk}\}$. For this purpose and later convenience, let us put

$$x_1 = 2(-su_2 + u_1u_3),$$
$$x_2 = 2(su_1 + u_2u_3),$$
$$x_3 = s^2 + u_3^2 - u_1^2 - u_2^2$$

and

$$y_1 = 2(-tv_2 + v_1v_3),$$
$$y_2 = 2(tv_1 + v_2v_3),$$
$$y_3 = t^2 + v_3^2 - v_1^2 - v_2^2.$$

Then $x_1^2 + x_2^2 + x_3^2 = 1$, $y_1^2 + y_2^2 + y_3^2 = 1$, moreover, when $U$ runs through the whole SU(2), the vector $(x_1, x_2, x_3)$ runs through the whole sphere, and similarly for the correspondence between $V$ and $(y_1, y_2, y_3)$. Also note the relations

$$\Pi_0^a \sigma_3 \Pi_0^a = \Pi_0^a, \qquad \Pi_1^a \sigma_3 \Pi_1^a = -\Pi_1^a,$$
$$\Pi_j^a \sigma_k \Pi_j^a = \mathbf{0}, \quad j = 0, 1; k = 1, 2,$$

and similar relations for $\Pi_k^b$. Now by use of Lemma 1 and the above relations, after direct but tedious algebraic manipulations, we obtain

$$p_{00} = \frac{1}{4}(1 + \vec{a}\vec{x} + \vec{b}\vec{y} + \vec{c}\vec{x}\vec{y}),$$
$$p_{01} = \frac{1}{4}(1 + \vec{a}\vec{x} - \vec{b}\vec{y} - \vec{c}\vec{x}\vec{y}),$$
$$p_{10} = \frac{1}{4}(1 - \vec{a}\vec{x} + \vec{b}\vec{y} - \vec{c}\vec{x}\vec{y}),$$
$$p_{11} = \frac{1}{4}(1 - \vec{a}\vec{x} - \vec{b}\vec{y} + \vec{c}\vec{x}\vec{y}).$$

Finally,

$$\mathcal{C}(\rho) = \sup_P \mathcal{I}(P(\rho)) = \sup_{U,V} I(p) = \sup_{\vec{x},\vec{y}} I(p)$$

which is the desired result.

As a consequence of Theorem 1, putting $\vec{a} = \vec{b} = 0$ and noting that

$$\sup_{\vec{x},\vec{y}} \vec{c}\vec{x}\vec{y} = \max\{|c_1|, |c_2|, |c_3|\},$$

we readily obtain the following result.

**Theorem 2** *Let*

$$\rho = \frac{1}{4}\left(\mathbf{1} + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j\right)$$

*and* $c = \max\{|c_1|, |c_2|, |c_3|\}$, *then*

$$\mathcal{C}(\rho) = \frac{1-c}{2}\log(1-c) + \frac{1+c}{2}\log(1+c).$$

It is interesting to further consider the particular case $c_1 = c_2 = c_3 = -c$. In this instance, the state $\rho$ turns out to be the Werner state [26]

$$\rho = (1-c)\frac{\mathbf{1}}{4} + c|\Psi^-\rangle\langle\Psi^-|, \quad c \in [0, 1]$$

with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and in particular, we have

$$\mathcal{C}(\rho) = \frac{1-c}{2}\log(1-c) + \frac{1+c}{2}\log(1+c).$$

## 3 Classical vs. Quantum Correlations

Recall that the quantum mutual information $\mathcal{I}(\rho)$ is a measure of total correlations, while $\mathcal{C}(\rho)$ can be interpreted as a measure of classical correlations. Thus we may regard the difference

$$\mathcal{Q}(\rho) = \mathcal{I}(\rho) - \mathcal{C}(\rho)$$

as a measure of quantum correlations. The natural question arises: what are the relationships among these three correlation measures. From the discussions in the introduction, we know that $\mathcal{C}(\rho) = \mathcal{Q}(\rho)$ for any pure bipartite state, and

$$\mathcal{C}(\rho) = I(p) \geq \mathcal{Q}(\rho) = 0$$

for any mixed state of the form

$$\rho = \sum_{jk} p_{jk} P_j^a \otimes P_k^b$$

where $p = \{p_{jk}\}$ is a bivariate probability distribution, $\{P_j^a\}$ and $\{P_k^b\}$ are orthogonal one-dimensional projections for parties $a$ and $b$, respectively. This state is actually a classical state expressed in the quantum formalism. With these observations, it is intuitive and natural to conjecture that

$$\mathcal{C}(\rho) \geq \mathcal{Q}(\rho),$$

which is actually an equivalent reformulation of the original Lindblad conjecture $\mathcal{C}(\rho) \geq \frac{1}{2}\mathcal{I}(\rho)$. By Theorem 2, we can demonstrate that the Lindblad conjecture is false.

To gain an intuitive feeling of the relationships between $\mathcal{C}(\rho)$ and $\mathcal{Q}(\rho)$, consider the state $\rho$ in Theorem 2, which has eigenvalues

$$\lambda_0 = \frac{1}{4}(1 - c_1 - c_2 - c_3),$$

$$\lambda_1 = \frac{1}{4}(1 - c_1 + c_2 + c_3),$$

$$\lambda_2 = \frac{1}{4}(1 + c_1 - c_2 + c_3),$$

$$\lambda_3 = \frac{1}{4}(1 + c_1 + c_2 - c_3).$$

The requirements $\lambda_j \in [0, 1]$ put natural constraints on the coefficients $c_j$. The marginal states are $\rho^a = \mathbf{1}/2$ and $\rho^b = \mathbf{1}/2$. Consequently, the quantum mutual information in $\rho$ is

$$\mathcal{I}(\rho) = 2 + \sum_{j=0}^{3} \lambda_j \log \lambda_j. \tag{2}$$

We define the difference

$$D(\rho) = \mathcal{Q}(\rho) - \mathcal{C}(\rho) = \mathcal{I}(\rho) - 2\mathcal{C}(\rho)$$

which, by Theorem 2 and (2), can be expressed as

$$D(\rho) = 2 + \sum_{j=0}^{3} \lambda_j \log \lambda_j - (1 - c)\log(1 - c) - (1 + c)\log(1 + c).$$

Without loss of generality, we may assume that

$$c = \max\{|c_1|, |c_2|, |c_3|\} = c_3 \in [0, 1]$$

is fixed, and consider $D(\rho)$ as a function of $c_1$ and $c_2$. The graphs of $D(\rho)$ versus $c_1$ and $c_2$ for $c = c_3 = 0.2, 0.4, 0.6, 0.8$ are depicted in Fig. 1, and the corresponding regions of $(c_1, c_2)$ such that $D(\rho)$ is positive are depicted in Fig. 2, with the black areas corresponding to $D(\rho) > 0$ at which the Lindblad conjecture is violated. Thus, the classical correlations, as measured by $\mathcal{C}(\cdot)$, can be smaller, as well as larger, than the quantum correlations, as measured by $\mathcal{Q}(\cdot)$.

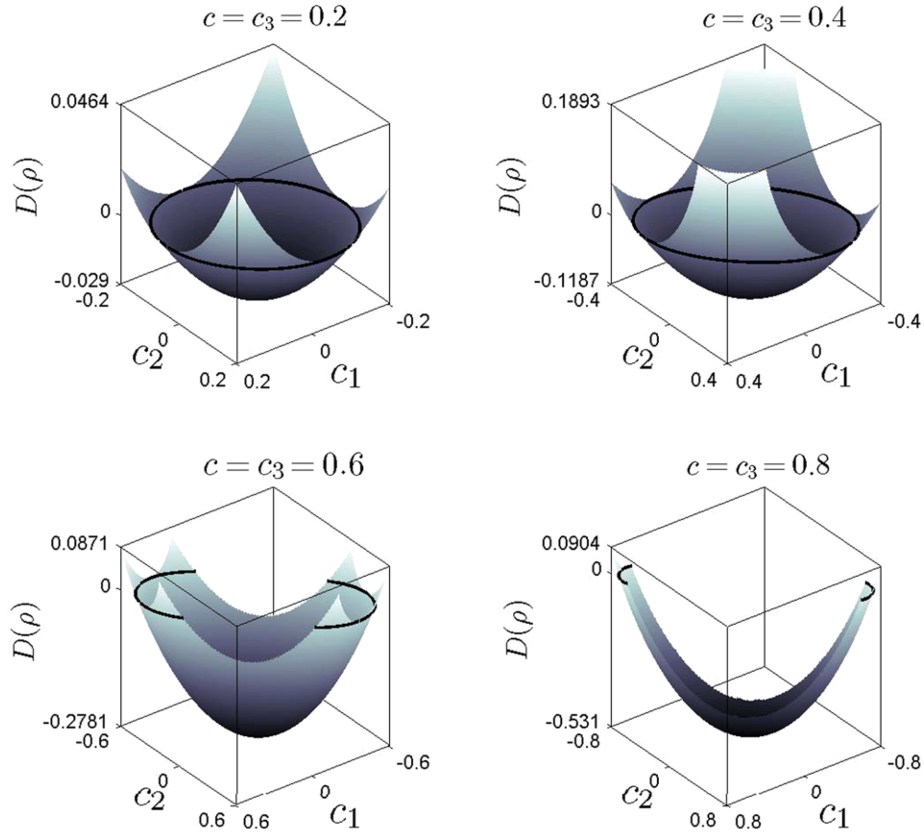


**Fig. 1** Graphs of $D(\rho)$ versus $c_1$ and $c_2$ for $c = c_3 = 0.2, 0.4, 0.6, 0.8$

## 4 Discussion

For characterizing quantum and classical correlations, there are various axiomatic and operational approaches, and it seems that no single measure can capture all the intrigue and subtle features of classical and quantum correlations. A plethora of entanglement measures are introduced and widely studied in the last decade in the entanglement/separability paradigm [26], with the purpose of quantifying quantum correlations related to nonlocality [4, 13, 27]. On the other hand, Henderson and Vedral initiated a different approach to quantifying classical correlations [12], while Olliver and Zurek introduced quantum discord as a measure of quantum correlations from the measurement perspective [19]. We now compare the observable correlations $\mathcal{C}(\rho)$ with the last two measures.

Consider the quantity defined by

$$\mathcal{C}_a(\rho) = \sup_{P^a} \mathcal{I}(P^a(\rho))$$

where $P^a(\rho) = \sum_j P_j^a \otimes \mathbf{1} \rho P_j^a \otimes \mathbf{1}$ and the supremum is over all one-side local measurement $P^a = \{P_j^a \otimes \mathbf{1}\}$. Let

$$\mathcal{Q}_a(\rho) = \mathcal{I}(\rho) - \mathcal{C}_a(\rho).$$

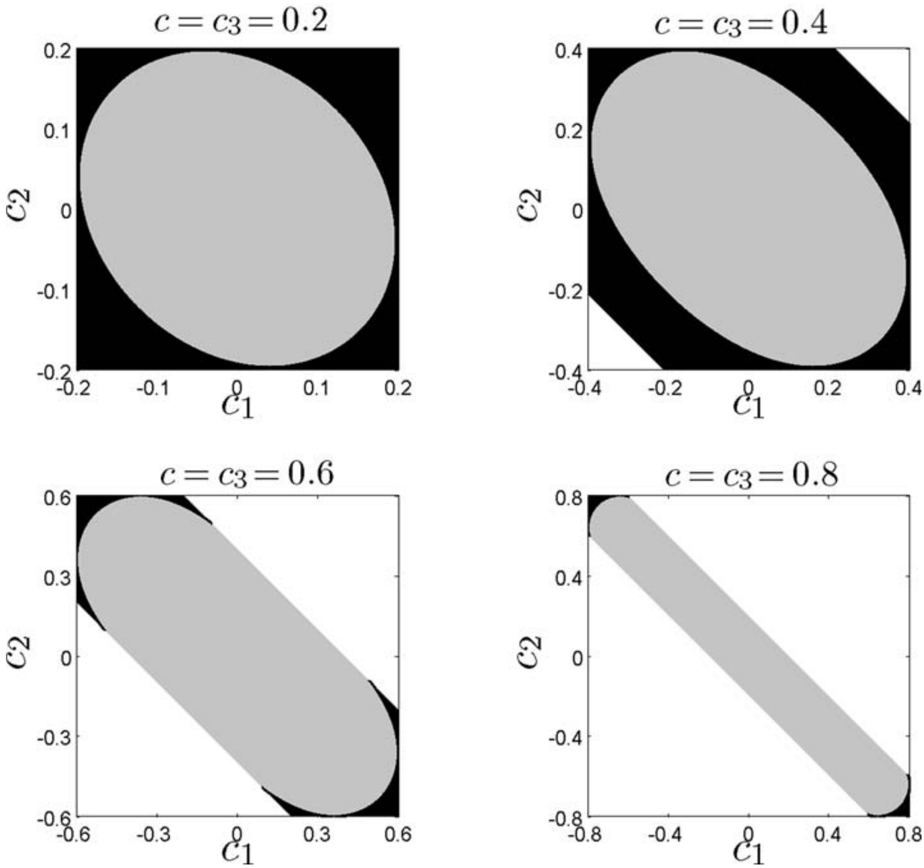


**Fig. 2** The *black regions* are the sets of $(c_1, c_2)$ such that $D(\rho) > 0$, the *grey regions* correspond to $D(\rho) < 0$

Then $\mathcal{C}_a(\rho)$ is closely related to a measure of classical correlations introduced by Henderson and Vedral [12] (the difference lies in that the latter allows general local measurements for party $a$), and $\mathcal{Q}_a(\rho)$ turns out to be the (minimum) quantum discord introduced by Ollivier and Zurek [19].

For any local von Neumann measurements $P^a = \{P^a_j\}$ and $P^b = \{P^b_k\}$, since the state

$$P(\rho) = \sum_{jk} P^a_j \otimes P^b_k \rho P^a_j \otimes P^b_k$$

can be regarded as the result of performing the local measurement $P^b = \{P^b_k\}$ on the state $P^a(\rho) = \sum_j P^a_j \otimes \mathbf{1}\rho P^a_j \otimes \mathbf{1}$, by the monotonicity of quantum relative entropy (which implies the monotonicity of quantum mutual information under local measurements) [18, 24], we have

$$\mathcal{I}(P^a(\rho)) \geq \mathcal{I}(P(\rho)).$$

Consequently, by taking supremum of both sides, we obtain

$$\mathcal{C}_a(\rho) \geq \mathcal{C}(\rho)$$

and thus

$$\mathcal{Q}_a(\rho) \leq \mathcal{Q}(\rho).$$

In particular, consider the state

$$\rho = \sum_j p_j |j\rangle\langle j| \otimes \rho_j^b$$

where $\{|j\rangle\}$ is an orthonormal base for party $a$. Then direct evaluation leads to $\mathcal{C}_a(\rho) = \mathcal{I}(\rho)$ and $\mathcal{Q}_a(\rho) = 0$, that is, the quantum discord views the above state as a "classical" state without any quantum correlations. However, unless the family of operators $\{\rho_j^b\}$ commute, it is impossible to identify $\rho$ as a bivariate classical probability distribution. The above state is actually a classical-quantum hybrid. From the viewpoint of correlations between the two parties, it is classical only in party $a$, and in general is quantum in party $b$, and there is no *a priori* reason to regard the overall correlations as classical. It is still a quantum object. In this sense, we say that the quantum discord underestimates quantum correlations since it fails to detect the quantum characteristics in such a state. Consequently, the measure $\mathcal{C}_a(\rho)$ overestimates the classical correlations. Another property of $\mathcal{C}_a(\rho)$ and $\mathcal{Q}_a(\rho)$ is that they are in general not symmetric with respect to the interchange of the two parties. With these observations, we may think of the observable correlations $\mathcal{C}(\rho)$ as a more natural measure of classical correlations than $\mathcal{C}_a(\rho)$, and $\mathcal{Q}(\rho)$ a more natural measure of quantum correlations than $\mathcal{Q}_a(\rho)$.

Finally, it should be remarked that in modern quantum information theory, a measurement is most generally represented by a POVM (positive-operator valued measure), and it is naturally and important to consider the observable correlations in such a more general setting (i.e., using POVMs rather than von Neumann measurements in the definition). This is a more difficult issue than the study of the accessible information for quantum ensembles. Since there have been seldom analytical results for the later issue in the last 40 years, we cannot hope there will be many analytical results for the former issue. Another problem is to consider the asymptotic cases of measurements in many copies. Due to the considerable complications caused by POVMs and the seemingly intractable optimization involved, we failed to obtain any analytical solutions, and numerical approach is required. We hope the readers will be more successful.

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## References

1. Adami, C., Cerf, N.J.: Von Neumann capacity of noisy quantum channels. Phys. Rev. A **56**, 3470–3483 (1997)
2. Barnett, S.M., Phoenix, S.J.D.: Entropy as a measure of quantum optical correlation. Phys. Rev. A **40**, 2404–2409 (1989)
3. Barnett, S.M., Phoenix, S.J.D.: Information theory, squeezing, and quantum correlations. Phys. Rev. A **44**, 535–545 (1991)
4. Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A **54**, 3824–3851 (1996)
5. Bhatia, R.: Matrix Analysis. Springer, Berlin (1997)

6. Cover, T.M., Thomas, J.A.: Elements of Information Theory. Wiley, New York (1991)
7. Cramér, H.: Mathematical Methods of Statistics. Princeton University Press, Princeton (1974)
8. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777–780 (1935)
9. Everett III, H.: The theory of the universal wavefunction. In: DeWitt, B.S., Graham, N. (eds.) The Many-Worlds Interpretation of Quantum Mechanics, pp. 3–140. Princeton University Press, Princeton (1973) (written in 1957, first publication herein)
10. Groisman, B., Popescu, S.A.: Winter, Quantum, classical, and total amount of correlations in a quantum state. Phys. Rev. A **72**, 032317 (2005)
11. Hall, M.J.W., Andersson, E., Brougham, T.: Maximum observable correlation for a bipartite quantum system. Phys. Rev. A **74**, 062308 (2006)
12. Henderson, L., Vedral, V.: Classical, quantum and total correlations. J. Phys. A **34**, 6899–6905 (2001)
13. Horodecki, M.: Entanglement measures. Quantum Inf. Comput. **1**, 3–26 (2001)
14. Li, N., Luo, S.: Total versus quantum correlations in quantum states. Phys. Rev. A **76**, 032327 (2007)
15. Lindblad, G.: Entropy, information and quantum measurements. Commun. Math. Phys. **33**, 305–322 (1973)
16. Lindblad, G.: Expectations and entropy inequalities for finite quantum systems. Commun. Math. Phys. **39**, 111–119 (1974)
17. Lindblad, G.: Quantum entropy and quantum measurements. In: Bendjaballah, C., et al.(eds.) Quantum Aspects of Optical Communications. Lecture Notes in Physics, vol. 378, pp. 71–80. Springer, Berlin (1991)
18. Nielsen, M., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
19. Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. **88**, 017901 (2002)
20. Peres, A.: Quantum Theory: Concepts and Methods. Kluwer Academic, Dordrecht (1998)
21. Schrödinger, E.: Discussion of probability relations between separated systems. Proc. Camb. Philos. Soc. **31**, 555–563 (1935)
22. Schumacher, B., Westmoreland, M.D.: Quantum mutual information and the one-time pad. Phys. Rev. A **74**, 042305 (2006)
23. Shannon, C.E., Weaver, W.: The Mathematical Theory of Communication. University of Illinois Press, Urbana (1949)
24. Vedral, V.: The role of relative entropy in quantum information theory. Rev. Mod. Phys. **74**, 197–234 (2002)
25. Wehrl, A.: General properties of entropy. Rev. Mod. Phys. **50**, 221–260 (1978)
26. Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys. Rev. A **40**, 4277–4281 (1989)
27. Wootters, W.K.: Entanglement of formation and concurrence. Quantum Inf. Comput. **1**, 27–44 (2001)